Analytic animated rings

§ Where are we going?

For an analytic adic space $X = \mathrm{Spa}(R, R^+)$,
we will define the analytic K-theory of X as

$$K^{an}(X) := K^{\mathrm{Efimov}}(\mathrm{Nuc}((R, R^+)_\blacksquare))$$

category of nuclear modules attached to analytic ring $(R, R^+)_\blacksquare$
(it is dualizable cat.)

Today We'll define (animated) analytic rings and define $(R, R^+)_\blacksquare$ in the case (R, R^+) is a discrete Huber pair.

§ Recap of Condensed Maths

(after Clausen-Scholze)

Def. A <u>condensed set</u>/<u>group</u>/<u>ring</u> --- is a sheaf of sets/groups/rings/-- on the proétale site of a geometric point $*_{\text{proét}}$.

Here, $*_{\text{proét}}$ has as objects profinite sets, and covers finite families of jointly surjective maps.

Given a topological space T one associates a condensed set $\underline{T} : \{\text{profinite sets}\}^{op} \longrightarrow \text{Sets}$

$$S \longmapsto C^0(S,T)$$

$T \mapsto \underline{T}$ defines a functor from topological spaces to condensed sets, which is fully faithful on "nice topological spaces" (e.g. metrizable spaces).

Recall that an <u>extremally disconnected set</u> S is a compact Hausdorff space such that any surjection $S' \twoheadrightarrow S$, from S' compact Hausdorff, splits.

<u>Fact</u> Extremally disc. sets form a basis for *proét.

<u>Rmk</u> Possible solutions to set theoretic issues:

1) Fix uncountable strong limit cardinal κ, and define κ-condensed sets $\mathrm{Cond}_\kappa\mathrm{Sets}$ (as above restricting to κ-small profinite sets).

2) $$\mathrm{Cond\,Sets} := \varinjlim_{\kappa} \mathrm{Cond}_\kappa\mathrm{Sets}$$
where the colim runs over uncountable strong limit cardinals κ.

3) Define <u>light condensed sets</u>
$$\mathrm{Cond\,Sets}^{\mathrm{light}}$$
restricting to metrizable profinite sets.

We will take 2) as a definition.

Similarly, for any category $\mathcal{C}$ admitting filtered colimits

$$\mathrm{Cond}(\mathcal{C}) := \varinjlim_{\kappa} \mathrm{Cond}_\kappa(\mathcal{C}), \text{ where } \mathrm{Cond}_\kappa(\mathcal{C}) := \text{functors}$$

$$\{\kappa\text{-small extr. disc. sets}\}^{op} \to \mathcal{C}$$

sending finite disjoint unions to finite products.

<u>Prop.</u> The category of condensed abelian groups Cond Ab is an abelian category satisfying same Grothendieck axioms as Ab. (but no non-zero injective objects).

Moreover, it is generated by the compact projective objects $\mathbb{Z}[\underline{S}]$, for varying extremally disconnected profinite sets S.

Here, for any condensed set X, we define the <u>free cond. abelian group</u>

$$\mathbb{Z}[X]$$

as the sheafification of $S \mapsto \mathbb{Z}[X(S)]$.

__Rmk__ The functor from topological ab. groups to cond. ab. groups commutes with limits, but not with colimits. E.g.

$$\underline{\mathbb{R}/\mathbb{Q}} \neq \underline{\mathbb{R}}/\underline{\mathbb{Q}}$$

it has indiscrete topology

new object, it "remembers" topology on $\mathbb{R}$.

There is a good notion of completion for cond. ab. groups, called __solidification.__

Let's try to understand $\mathbb{Z}[S]$ for S profinite set.

For S finite set,

$$\mathbb{Z}[S] = \bigcup_n \mathbb{Z}[S]_{\leq n}$$

where $\mathbb{Z}[S]_{\leq n} := \left\{ \sum_{s \in S} n_s[s] \;\middle|\; \sum_{s \in S} |n_s| \leq n \right\} \subset \mathbb{Z}[S]$.

Fact For $S = \varprojlim_n S_i$, with S_i finite,

$$\mathbb{Z}[S] = \bigcup_n \varprojlim_i \mathbb{Z}[S_i]_{\leq n} \subset \varprojlim_i \mathbb{Z}[S_i].$$

Def For $S = \varprojlim_i S_i$ profinite, with S_i finite,

$$\mathbb{Z}[S]^{\blacksquare} := \varprojlim_i \mathbb{Z}[S_i] \quad \left(\text{"completion" of } \mathbb{Z}[S]\right)$$

(Idea: want to "enlarge" $\mathbb{Z}[S]$ allowing more "convergent sums")

Def $M \in \text{CondAb}$ is <u>solid abelian group</u> if $\forall S$ profinite

$$\begin{tikzcd} \mathbb{Z}[S] \arrow[r, "\forall f"] \arrow[d] & M \\ \mathbb{Z}[S]^{\blacksquare} \arrow[ur, dashed, "\exists! \tilde{f}"'] & \end{tikzcd}$$

In other words, $\forall S$ profinite

$$\text{Hom}_{\mathbb{Z}}(\mathbb{Z}[S]^{\blacksquare}, M) \longrightarrow \text{Hom}_{\mathbb{Z}}(\mathbb{Z}[S], M).$$

is an isomorphism.

Thm 1) The category

$$\mathrm{Solid} \subset \mathrm{CondAb} \quad (*)$$

of solid abelian groups is an ab. cat. stable under limits, colimits and extensions.

2) The inclusion $(*)$ admits a left adjoint

$$\mathrm{CondAb} \to \mathrm{Solid} : M \mapsto M^{\blacksquare}$$

Taking $\mathbb{Z}[S]$ to $\mathbb{Z}[S]^{\blacksquare}$.

3) The obj. $\mathbb{Z}[S]^{\blacksquare}$, for varying S extr. disc. set, are compact projective generators of Solid; $\mathbb{Z}[S]^{\blacksquare} \cong \prod_I \mathbb{Z}$, for some I.

The obj. $\prod_I \mathbb{Z}$, for varying I, are the compact proj. obj. of Solid.

4) $\forall C \in D(\mathrm{Solid})$, $\forall S$ profinite, $R\underline{\mathrm{Hom}}(\mathbb{Z}[S]^{\blacksquare}, C) \xrightarrow{\sim} R\underline{\mathrm{Hom}}(\mathbb{Z}[S], C)$.

Examples 1) For any A discrete ab. group, $\underline{A} \in \mathrm{Solid}$

2) $\mathbb{Z}_p = \varprojlim_n \mathbb{Z}/p^n \in \mathrm{Solid}$, $\mathbb{Q}_p = \mathbb{Z}_p[\frac{1}{p}] \in \mathrm{Solid}$.

3) For any $\mathbb{Q}_p$-Banach space V, $V \cong \left(\bigoplus_I \mathbb{Z}_p\right)^{\wedge}_p [\frac{1}{p}] \in \mathrm{Solid}$.

For $M, N \in \mathrm{Solid}$,

$$M \otimes^{\blacksquare} N := (M \otimes N)^{\blacksquare}.$$

This defines a symmetric monoidal tensor structure on Solid.

<u>Fact</u> $\left(\prod_I \mathbb{Z}\right) \otimes^{\blacksquare} \left(\prod_J \mathbb{Z}\right) = \prod_{I \times J} \mathbb{Z}$, for any I, J.

<u>Example</u> $\mathbb{Z}[[U]] \otimes^{\blacksquare} \mathbb{Z}[[T]] = \mathbb{Z}[[U, T]]$

---

§ <u>Analytic rings</u>

Let A a condensed associative ring (all rings will be unital).

Want define notion of "completion" for

$$\mathrm{Mod}_A^{\mathrm{cond}} := \mathrm{Mod}_A(\mathrm{CondAb}).$$

measures/modules

Def An <u>analytic ring</u> is a pair $(A, \mathcal{M})$ where A is cond. ass. ring, and $\mathcal{M}$ is functor

$$\mathcal{M} : \mathrm{ExDisc} \to \mathrm{Mod}_A^{\mathrm{cond}} : S \mapsto \mathcal{M}[S]$$

extremally disc. sets

Taking finite disjoint unions to finite products, together with a natural transformation $S \longrightarrow \mathcal{M}[S]$, satisfying following property: for any complex

$$C: \cdots \to C_i \to \cdots \to C_1 \to C_0 \to 0$$

with $C_i \in \mathrm{Mod}_A^{\mathrm{cond}}$ which is direct sum of $\mathcal{M}[S]$'s for $S \in \mathrm{ExDisc}$, the map

$$R\underline{\mathrm{Hom}}_A(\mathcal{M}[S'], C) \longrightarrow R\underline{\mathrm{Hom}}_A(A[S'], C)$$

is an isom. for all $S' \in \mathrm{ExDisc}$.

Here, for $M, N \in \mathrm{Mod}_A^{\mathrm{cond}}$, $R\underline{\mathrm{Hom}}(M,N)$ is def. by

$$R\underline{\mathrm{Hom}}(M,N)(S) = R\mathrm{Hom}(M \otimes_{\mathbb{Z}} \mathbb{Z}[S], N)$$

for $S \in \mathrm{ExDisc}$.

Prop. Let $(A, \mathcal{M})$ be an analytic ring.

1) The full subcat

$$\mathrm{Mod}_{(A,\mathcal{M})}^{\mathrm{cond}} \subset \mathrm{Mod}_A^{\mathrm{cond}} \quad (*)$$

"$(A,\mathcal{M})$-complete modules"

of A-mod. M s.t. $\forall S \in \mathrm{ExDisc}$

$$\mathrm{Hom}_A(\mathcal{M}[S], M) \to \mathrm{Hom}_A(A[S], M)$$

is an isom., is an abelian cat stable under limits, colimits, extensions.

The objects $\mathcal{M}[S]$, $S \in \mathrm{ExDisc}$, form family of compact projective generators.

The inclusion $(*)$ admits a left adjoint

$$\mathrm{Mod}_A^{\mathrm{cond}} \to \mathrm{Mod}_{(A,\mathcal{M})} : M \mapsto \text{"}M \otimes_A (A,\mathcal{M})\text{"}.$$

sending $A[S]$ to $\mathcal{M}[S]$, for $S \in \mathrm{ExDisc}$.

2) The functor
$$\mathcal{D}(\mathrm{Mod}_{(A,\mathcal{M})}^{\mathrm{cond}}) \to \mathcal{D}(\mathrm{Mod}_A^{\mathrm{cond}})$$

is fully faithful with ess. image stable under limits and colimits, and given by $C \in \mathcal{D}(\mathrm{Mod}_A^{\mathrm{cond}})$ s.t. $\forall S \in \mathrm{ExDisc}$.

$$R\mathrm{Hom}_A(\mathcal{M}[S], C) \to R\mathrm{Hom}_A(A[S], C)$$

is an isom; in this case also $\underline{R\mathrm{Hom}}_A$'s agree.

<u>Examples</u> (of analytic rings)

1) $\mathbb{Z}_\blacksquare := (\underline{\mathbb{Z}}, \mathcal{M}_{\mathbb{Z}})$ with

$$\mathcal{M}_{\mathbb{Z}} : \mathrm{ExDisc} \to \mathrm{CondAb} : S \mapsto \mathbb{Z}[S]^\blacksquare$$

is an analytic ring.

2) For any discrete ring A,

$(A, \mathbb{Z})_\blacksquare := (\underline{A}, \mathcal{M}_A)$ with

$$\mathcal{M}_A : \mathrm{ExDisc} \to \mathrm{Mod}_A^{\mathrm{Cond}} : S \mapsto \mathbb{Z}[S]^\blacksquare \otimes_{\mathbb{Z}} \underline{A}$$

is an analytic ring.

In fact, as $\mathcal{M}_A[S] = \mathbb{Z}[S]^\blacksquare \otimes^L_{\mathbb{Z}} \underline{A}$ (since $\mathbb{Z}[S]^\blacksquare \simeq \prod_I \mathbb{Z}$ and A is discrete), this follows formally by tensor-hom adjunction from 1).

(This suggests we want to "derive" our rings to have "base change property" for analytic rings)

<u>Thm</u> For any fin. gen. $\mathbb{Z}$-alg., $A_\blacksquare := (A, \mathcal{M}_A)$ with

$$\mathcal{M}_A : \mathrm{ExDisc} \to \mathrm{Mod}_A^{\mathrm{Cond}} : S = \varprojlim_i \underset{\text{finite}}{S_i} \mapsto \varprojlim_i A[S_i]$$

is an analytic ring.

(for $S \in \mathrm{ExDisc}$, $\mathcal{M}(S) \cong \prod_I A$, for some set I).

__Cor.__ For any discrete ring A,

$$A_\blacksquare := \operatorname*{colim}_{\substack{A' \to A \\ A' \text{ f.g. } \mathbb{Z}\text{-alg}}} A'_\blacksquare \quad \text{is an analytic ring.}$$

§ Analytic animated rings

__Def__ Let $\mathcal{C}$ cat. having all small colimits. Denote $\mathcal{C}^{cp} \subset \mathcal{C}$ full subcat of compact projective objects. Assume $\mathcal{C}$ is generated under small colimits by $\mathcal{C}^{cp}$.

The __animation__ of $\mathcal{C}$ is the ∞-cat. $\mathrm{Ani}(\mathcal{C})$ freely gen. under sifted colimits by $\mathcal{C}^{cp}$.

__Examples__ 1) $\mathcal{C} = \mathrm{Set}$, $\mathcal{C}^{cp} = \{\text{finite sets}\}$.

$$\mathrm{Ani}(\mathcal{C}) =: \mathrm{Ani}$$

is the ∞-cat of __anima__ (or spaces).

More concretely, sSet by inverting weak equivalences.

2) $C = \mathrm{Ab}$, $C^{cp} = \{\text{finite free abelian groups}\}$.

By Dold-Kan equivalence,

$$\mathrm{Ani}(\mathrm{Ab}) \simeq D_{\geq 0}(\mathrm{Ab}).$$

3) $C = \mathrm{CondSet}$, $C^{cp} = \mathrm{ExDisc}$

$\mathrm{Ani}(\mathrm{CondSet})$ animated cond. sets.

4) $C = \mathrm{CondAb}$, $C^{cp} = \left\{ \begin{array}{l} \text{direct summands of } \mathbb{Z}[S], \\ \text{for } S \in \mathrm{ExDisc} \end{array} \right\}$.

$$\mathrm{Ani}(\mathrm{CondAb}) \simeq D_{\geq 0}(\mathrm{CondAb}).$$

5) $C = \mathrm{CondRing}$, $C^{cp} = \left\{ \begin{array}{l} \text{retracts of } \mathbb{Z}[\mathbb{N}[S]], \\ \text{for } S \in \mathrm{ExDisc} \end{array} \right\}$

<u>Fact</u> C gen. under small colimits by C^{cp}.

There is a natural equiv. of ∞-categories

$$\mathrm{Cond}(\mathrm{Ani}(C)) \simeq \mathrm{Ani}(\mathrm{Cond}(C)).$$

gen under small colims by comp. proj. obj.

Def An <u>analytic</u> <u>animated</u> <u>associative ring</u> is a pair $(A, \mathcal{M})$ where A is animated ass. cond. ring, and

$$\mathcal{M} : \mathrm{ExDisc} \longrightarrow D_{\geq 0}(A) : S \longmapsto \mathcal{M}[S]$$

is functor taking finite coproducts to finite direct sums, together with a natural transformation $S \to \mathcal{M}[S]$ of condensed anima, satisfying following property: for any $C \in D_{\geq 0}(A)$ that is sifted colimit of objects of the form $\mathcal{M}[S]$, the map

$$\underline{\mathrm{Hom}}_{D_{\geq 0}(A)}(\mathcal{M}[S'], C) \to \underline{\mathrm{Hom}}_{D_{\geq 0}(A)}(A[S'], C)$$

of condensed anima is an equiv. $\forall S' \in \mathrm{ExDisc}$.

Def. Let $(A, \mathcal{M})$ analytic animated associative ring. We def.

$$D_{\geq 0}(A, \mathcal{M}) \subset D_{\geq 0}(A)$$

as the full ∞-subcat. spanned by all $C \in D_{\geq 0}(A)$ s.t.

$\forall S \in \mathrm{ExDisc}$, the map

$$\underline{\mathrm{Hom}}_{D_{\geq 0}(A)}(M[S], C) \to \underline{\mathrm{Hom}}_{D_{\geq 0}(A)}(A[S], C)$$

of cond. anima is an equivalence.

<u>Prop.</u> Let (A, M) be analytic animated ass. ring.

1) The ∞-cat $D_{\geq 0}(A, M)$ is gen. under sifted colimits by $M[S]$, for varying $S \in \mathrm{ExDisc}$, which are compact proj. objects of $D_{\geq 0}(A, M)$. The full ∞-subcat.

$$D_{\geq 0}(A, M) \subset D_{\geq 0}(A)$$

is stable under all limits and colimits, and admits a left adjoint

$$- \otimes_A (A, M) : D_{\geq 0}(A) \to D_{\geq 0}(A, M)$$

sending $A[S]$ to $M[S]$.

2) The ∞-cat. $D_{\geq 0}(A, M)$ is prestable. Its heart is the ab. cat $D^{\heartsuit}(A, M)$ that is the full subcat of $\mathrm{Mod}^{\mathrm{cond}}_{\pi_0 A}$ gen. under colimits by $\pi_0 M[S]$, for $S \in \mathrm{ExDisc}$.

An object $C \in D_{\geq 0}(A)$ lies in $D_{\geq 0}(A, M)$ iff $H_i(C) \in D^{\heartsuit}(A, M)\ \forall i$.

If A is cond. animated comm. ring, there is a unique symm. monoidal structure $- \otimes_{(A,M)} -$ on $D_{\geq 0}(A, M)$ making $- \otimes_A (A, M)$ symm monoidal

Def. A map of analytic animated ass. rings $(A, M) \to (B, N)$ is a map $A \to B$ of cond. animated ass. rings s.t. the forgetful functor $D_{\geq 0}(B, N) \to D_{\geq 0}(A)$ takes image in $D_{\geq 0}(A, M)$.

Prop. Let $f : (A, M) \to (B, N)$ be a map of analytic animated ass. rings. The forgetful functor $D_{\geq 0}(B, N) \to D_{\geq 0}(A, M)$ admits a left adjoint

$$- \otimes_{(A,M)} (B, N) : D_{\geq 0}(A, M) \to D_{\geq 0}(B, N)$$

sending $M[S]$ to $N[S]$.

! The following result only holds in the animated setting!

Prop. Let (A, M) analytic animated ass. ring, $g : A \to B$ a map of cond. animated ass. rings. The functor $N : \mathrm{ExDisc} \to D_{\geq 0}(B) : S \mapsto N[S] := B[S] \otimes_A (A, M)$ defines analytic animated ass. ring (B, N).

Proof (In the case A is commutative).

WTS For any $C \in D_{\geq 0}(B)$ sifted colim. of $N(T)$ for $T \in \mathrm{ExDisc}$

$$\underline{\mathrm{Hom}}_{D_{\geq 0}(B)}(N(S), C) \xrightarrow{\sim} \underline{\mathrm{Hom}}_{D_{\geq 0}(B)}(B(S), C), \forall S \in \mathrm{ExDisc}.$$

As $C \in D_{\geq 0}(A, M)$, we have

$$\underline{\mathrm{Hom}}_{D_{\geq 0}(A)}(N[S], C) = \underline{\mathrm{Hom}}_{D_{\geq 0}(A)}(B[S], C)$$

$$\Rightarrow \underline{\mathrm{Hom}}_{D_{\geq 0}(B)}(N[S] \otimes_A B, C) = \underline{\mathrm{Hom}}_{D_{\geq 0}(B)}(B[S] \otimes_A B, C)$$

$$\Rightarrow \underline{\mathrm{Hom}}_{D_{\geq 0}(B)}(N[S] \otimes_A M, C) = \underline{\mathrm{Hom}}_{D_{\geq 0}(B)}(B[S] \otimes_A M, C)$$

$\underline{\mathrm{Hom}}_{D_{\geq 0}(B)}(M, -)$

$M \in D_{\geq 0}(B)$

$$\Leftrightarrow \underline{\mathrm{Hom}}_{D_{\geq 0}(B)}(N(S) \otimes_B (B \otimes_A M), C) = \underline{\mathrm{Hom}}_{D_{\geq 0}(B)}(B[S] \otimes_B (B \otimes_A M), C).$$

The objects of the form $B \otimes_A M \in D_{\geq 0}(B)$ for $M \in D_{\geq 0}(B)$ generate all $D_{\geq 0}(B)$ under colimits; consider resolution of B

$$\cdots \to B \otimes_A B \otimes_A B \to B \otimes_A B \to B.$$

This implies the statement. ∎

Prop. Let (A, A^+) pair of discrete comm. rings with $A^+ \subset A$. The animated analytic ring $(A, A^+)_\blacksquare := (A, M_{A^+})$ where

$$M_{A^+} : \text{ExDisc} \to D_{\geq 0}(A) : S \mapsto A(S) \otimes^L_{A^+} A^+_\blacksquare$$

is 0-truncated and

$$A(S) \otimes^L_{A^+} A^+_\blacksquare \simeq A \otimes_{A^+} A^+_\blacksquare[S].$$

Proof sketch Note that A is $A^+_\blacksquare$-complete: as A is discrete, it can be written as colim of copies of A^+ (which is $A^+_\blacksquare$-compl.). Therefore,

$$A(S) \otimes^L_{A^+} A^+_\blacksquare \simeq (A \otimes^L_{A^+} A^+(S)) \otimes^L_{A^+} A^+_\blacksquare$$

$$\simeq A \otimes^L_{A^+_\blacksquare} A^+_\blacksquare(S).$$

So, it remains to note that $A \otimes^L_{A^+} A^+_\blacksquare(S)$ is conc. in deg 0, and $A^+_\blacksquare$-complete. $\blacksquare$

Fact Given (A, A^+) pair of discrete comm. rings with $A^+ \subset A$. We have $(A, A^+)_\blacksquare \simeq (A, \overline{A^+})_\blacksquare$

where $\overline{A}^+ \subset A$ integral closure of A^+ in A.

Def. A discrete Huber pair is a pair (A, A^+) of discrete comm. rings with $A^+ \subset A$ integrally closed.

§ Alternative characterization of analytic rings

Prop. Let $\mathcal{A}$ be a condensed animated associative ring. A full sub-∞-cat. $\mathcal{D} \subset \mathcal{D}_{\geq 0}(\mathcal{A})$ is of the form $\mathcal{D}_{\geq 0}(\mathcal{A}, \mathcal{M})$ for a necessarily unique analytic ring structure $(\mathcal{A}, \mathcal{M})$ on $\mathcal{A}$ iff it satisfies the following conditions:

(1) The subcat $\mathcal{D} \subset \mathcal{D}_{\geq 0}(\mathcal{A})$ is stable under limits and colimits.

(2) The subcat. $\mathcal{D} \subset \mathcal{D}_{\geq 0}(\mathcal{A})$ is stable under $\underline{\mathrm{Hom}}_{\mathcal{D}_{\geq 0}(\mathrm{CondAb})}(M, -)$, for any $M \in \mathcal{D}_{\geq 0}(\mathrm{CondAb})$.

(3) The inclusion $\mathcal{D} \subset \mathcal{D}_{\geq 0}(\mathcal{A})$ admits a left adjoint.

(automatic up to set-theoretic issues)

<u>Proof</u> For the forward direction (1), (3) clear, and for (2) one can reduce to checking the stability under

$$\underline{\mathrm{Hom}}_{D_{\geq 0}(\mathrm{Cond}\,\mathrm{Ab})}(\mathbb{Z}[S], -), \text{ for any } S \in \mathrm{ExDisc}.$$

Conversely, we def. $M[S]$ as the image of $A[S]$ under the left adjoint from (3). Then, $\forall C \in D$, $\forall S \in \mathrm{ExDisc}$,

$$\mathrm{Hom}_{D_{\geq 0}(A)}(M[S], C) \xrightarrow{\sim} \mathrm{Hom}_{D_{\geq 0}(A)}(A[S], C)$$

is an isom. of anima, and by (2) we also have an isom. of condensed anima replacing Hom with $\underline{\mathrm{Hom}}$: in fact, $\forall T \in \mathrm{ExDisc}$,

$$\mathrm{Hom}_{D_{\geq 0}(A)}(A[T], \underline{\mathrm{Hom}}_{D_{\geq 0}(A)}(M[S], C))$$

$$= \mathrm{Hom}_{D_{\geq 0}(A)}(M[S], \underline{\mathrm{Hom}}_{D_{\geq 0}(A)}(A[T], C))$$

$$\overset{(2)}{=} \mathrm{Hom}_{D_{\geq 0}(A)}(A[S], \underline{\mathrm{Hom}}_{D_{\geq 0}(A)}(A[T], C))$$

$$= \mathrm{Hom}_{D_{\geq 0}(A)}(A[T], \underline{\mathrm{Hom}}_{D_{\geq 0}(A)}(A[S], C)).$$

Lastly, by (1), $\mathcal{D}$ contains all sifted colimits of $\mathcal{M}[S]$'s, so we are done. ■