DAG SEMINAR, PROBLEM SET 3 (OCT. 10-17).

Solving this PS necessitates reading Sect. 2.1-2.2 in the book

1. Let $\mathcal{C}$ be an ordinary category. Consider the 2-category $\{\text{Grpds}\}^{\mathcal{C}^{op}}$ of functors $\mathcal{C}^{op} \to \{\text{Groupoids}\}$. Consider the 2-category $\text{Cats}/\mathcal{C}$ of categories over $\mathcal{C}$ (we only allow equivalences between categories over $\mathcal{C}$ and natural transformations that are isomorphisms). Note that both sides are actually 2-groupoids (i.e., all 1-morphisms and 2-morphisms are invertible).

(a) Construct a 2-functor $\{\text{Grpds}\}^{\mathcal{C}^{op}} \to \text{Cats}/\mathcal{C}$, and show that it's *fully faithful*[1].

(b) Show that the essential image of the functor in (a) consists of those $p : \mathcal{C}' \to \mathcal{C}$, such that for any $\mathbf{c}' \in \mathcal{C}'$, the functor p induces an equivalence $\mathcal{C}'_{/\mathbf{c}'} \to \mathcal{C}_{/p(\mathbf{c}')}$.

2. Let S be a simplicial set and let $p : X \to S$ be a left fibration.

(a) Show that for any $s \in S$, the simplicial set X_s is a Kan simplicial set (cf. PS 1, Problem 1).

(b) For every edge $e : s_1 \to s_2$ in S, construct the map $e_! : X_{s_1} \to X_{s_2}$ well-defined up to homotopy.

(c) Show that the assignment $s \mapsto X_s$, $(e : s_1 \to s_2) \mapsto e_!$ defines a functor from the fundamental groupoid of S to the homotopy category of Kan simplicial sets.

(d) Assume that S is Kan. Show that in this case the maps $e_! : X_{s_1} \to X_{s_2}$ are homotopy equivalences and that p is a Kan fibration.

(e) The previous point can be strengthened: namely, we have the theorem that says that for any S and a left fibration $p : X \to S$, p is a Kan fibration if and only if the maps $X_{s_1} \to X_{s_2}$ are homotopy equivalences. Verify the Kan condition on simplices of dimensions 0,1, and 2.

3. We say that a map of simplicial sets $A \to A'$ is *left (resp., inner; right) anodyne* if it belongs to the class of maps generated by taking push-outs and retracts [2] by the class of inclusions $\Lambda^n_i \hookrightarrow \Delta^n$ for $i \neq n$ (resp., $i \neq 0, n$; $i \neq n$).

(a) Show that if a map is left/right anodyne, then it has a left lifting property with respect to left/right fibrations, defined as in Defn. 2.0.0.3. Show that if a map is inner anodyne, then any map $A \to X$, where X is a quasi-category can be extended to a map $A' \to X$.

NB: There is a general theorem (referred to as "the small object argument", see Prop. A.1.2.5) that says a map $A \to A'$ is left/right anodyne *if and only if* it has a left lifting property with respect to all left/right fibrations.

(b) Show that if $A \to A'$ is an arbitrary monomorphism, and $B \to B'$ is left anodyne, then

$$(A * B') \bigsqcup_{A*B} (A' * B) \hookrightarrow A' * B'$$

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[1]When talking about a 2-functor F between two 2-categories $\mathcal{D}_1 \to \mathcal{D}_2$, we say that it's fully faithful if it induces equivalences between Hom categories.

[2]plus certain colimits, see Defn. A.1.2.2

is inner anodyne.

(c) Deduce that if X is a quasi-category, and $f : K \to X$ be an arbitrary map of simplicial sets, then $X_{f/} \to X$ is a left fibration.

4. (a) There is a theorem that says that if $A \to A'$ is left anodyne, and $B \to B'$ is an arbitrary embedding, then

$$(A \times B') \underset{A \times B}{\sqcup} (A' \times B) \hookrightarrow A' \times B'$$

is left anodyne. Argue that it's enough to prove this for $A \to A'$ being $\Lambda^n_i \hookrightarrow \Delta^n$ for $i \neq n$ and $B \to B'$ being $\partial\Delta^m \hookrightarrow \Delta^m$. Verify that the latter maps are indeed left anodyne for small values of n and m.

(b) Deduce that a map $p : X \to S$ is a left fibration in the sense of Defn 2.0.0.3., then the induced map

$$\mathrm{Maps}(\Delta^1, X) \to X \underset{S}{\times} \mathrm{Maps}(\Delta^1, S)$$

is a trivial Kan fibration (this is the definition of left fibration given during the lecture). Here $\mathrm{Maps}(\Delta^1, X) \to X = \mathrm{Maps}(\Delta^0, X)$ is given by evaluation on the 0-vertex, and similarly for the map $\mathrm{Maps}(\Delta^1, S) \to S$.

(c) Let $p : X \to S$ be a left fibration. For an arbitrary simplicial set T/S we can form the simplicial set $\mathrm{Maps}_S(T, X)$. Show that it is Kan.

(*d) Strengthen (c) to show that if a map $X \to Y$ between left fibrations is a *pointwise equivalence* over S, then it admits an inverse up to homotopy.

5. Let S be a simplicial set. Consider the the (ordinary) category $(\mathcal{S}et_\Delta)^{\mathfrak{C}[S]^{op}}$ of simplicial functors $\mathfrak{C}[S]^{op} \to \mathcal{S}et_\Delta$ and the (ordinary) category $\mathcal{S}et_\Delta/S$.[3] In the lecture we defined the *unstraightening* functor $Un : (\mathcal{S}et_\Delta)^{\mathfrak{C}[S]^{op}} \to \mathcal{S}et_\Delta/S$.

(a) For an object $F \in (\mathcal{S}et_\Delta)^{\mathfrak{C}[S]^{op}}$ describe explicitly the simplices of the simplicial set $Un(F)$.

(b) Assume that F takes values in the subcategory $Kan \subset \mathcal{S}et_\Delta$. Show that the resulting object $Un(F) \to S$ is a right fibration.

6. Let $S \in \mathcal{S}et_\Delta$ and Un be as above. It is easy to see that the functor Un admits a left adjoint, denoted St and called the *straightening* functor. For a vertex $s \in S$; consider the simplicial set Δ^0 mapping to S via s; denote the resulting object of $\mathcal{S}et_\Delta/S$ by $\{s\}$. Consider the corresponding simplicial functor $St(\{s\}) : \mathfrak{C}[S]^{op} \to \mathcal{S}et_\Delta$. Show that it's isomorphic to the Yoneda functor $\mathrm{Hom}_{\mathfrak{C}[S]}(-, s)$.

7. Let $\mathcal{C}$ be a model category. Recall that two maps $f_1, f_2 \in \mathrm{Hom}_{\mathcal{C}}(x, y)$ are called homotopic if there exists a cylinder object $x \sqcup x \to C(x)$ and a map $F : C(x) \to y$ that restricts to $f_1 \sqcup f_2$. Assume now that x is cofibrant and y is fibrant. Show that homotopy is an equivalence relation on $\mathrm{Hom}_{\mathcal{C}}(x, y)$ and that the quotient set canonically identifies with $\mathrm{Hom}_{\mathrm{Ho}(\mathcal{C})}(x, y)$.

[3] Both of these are naturally simplicial categories, but we consider them as ordinary categories by taking 0-simplices of the Hom sets.